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2008 J. Phys. A: Math. Theor. 41 475206

(http://iopscience.iop.org/1751-8121/41/47/475206)

J. Phys. A: Math. Theor. **41** (2008) 475206 (27pp) doi:10.1088/1751-8113/41/47/475206

Löwner equations, Hirota equations and reductions of the universal Whitham hierarchy

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Received 11 August 2008, in final form 19 September 2008
Published 20 October 2008
Online at stacks.iop.org/JPhysA/41/475206

Abstract
This paper reconsiders finite variable reductions of the universal Whitham hierarchy of genus zero in the perspective of dispersionless Hirota equations. In the case of one-variable reduction, dispersionless Hirota equations turn out to be a powerful tool for understanding the mechanism of reduction. All relevant equations describing the reduction (Löwner-type equations and diagonal hydrodynamic equations) can be thereby derived and justified in a unified manner. The case of multi-variable reductions is not so straightforward. Nevertheless, the reduction procedure can be formulated in a general form, and justified with the aid of dispersionless Hirota equations. As an application, previous results of Guil, Mañas and Martínez Alonso are reconfirmed in this formulation.

PACS number: 02.30.Ik
Mathematics Subject Classification: 35Q58, 37K10

1. Introduction

The Löwner equation [1] is a differential equation that describes a family of deformations (with parameter λ) of univalent conformal maps $g_\lambda : D \to B_\lambda$ from a fixed disc $D \subset \mathbf{CP}^1$ to a simply connected domain $B_\lambda = B \setminus \Gamma_\lambda \subset \mathbf{CP}^1$ with a slit formed by a continuously growing arc Γ_λ on a fixed curve Γ. g_λ is the inverse of a univalent conformal map $f_\lambda : B_\lambda \to D$ whose existence is ensured by Riemann's mapping theorem. For technical reasons, we assume that D is centered at ∞, B contains ∞ in its interior, and $g_\lambda(\infty) = \infty$. To write the Löwner equation, let us express the maps g_λ and f_λ in terms of coordinates as $z = g(p, \lambda)$ and $p = f(z, \lambda)$. The Löwner equation thereby reads

$$\frac{\partial f(z,\lambda)}{\partial\lambda} = f(z,\lambda)\frac{\kappa(\lambda) + f(z,\lambda)}{\kappa(\lambda) - f(z,\lambda)}\frac{\partial\phi(\lambda)}{\partial\lambda}$$

for $f(z,\lambda)$ and

$$\frac{\partial g(p,\lambda)}{\partial\lambda} = p\frac{p+\kappa(\lambda)}{p-\kappa(\lambda)}\frac{\partial g(p,\lambda)}{\partial p}\frac{\partial\phi(\lambda)}{\partial\lambda}$$

for $g(p,\lambda)$. (As one can easily see, these two equations are equivalent.) $\kappa(\lambda)$ and $\phi(\lambda)$ are auxiliary functions that are determined by (and, conversely, determine) the deformations of the maps. $\kappa(\lambda)$ is called a 'driving force', and $\phi(\lambda)$ is related to the behavior of $g(p,\lambda)$ and $f(z,\lambda)$ at infinity:

$$\begin{aligned} g(p,\lambda) &= \mathrm{e}^{\phi(\lambda)}p + O(1) \qquad (p\to\infty),\\ f(z,\lambda) &= \mathrm{e}^{-\phi(\lambda)}z + O(1) \qquad (z\to\infty). \end{aligned}$$

A few variants of the Löwner equation are also known. Firstly, one can consider a family of univalent conformal maps $g_{\boldsymbol{\lambda}}$ to a domain $B_{\boldsymbol{\lambda}}$ with multiple slits of arcs on several fixed curves $\Gamma_1,\ldots,\Gamma_M$. Since the arcs can grow independently, this family depends on M-dimensional parameters $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_M)$. The univalent homomorphic functions $g(p,\boldsymbol{\lambda})$ and $f(z,\boldsymbol{\lambda})$ representing $g_{\boldsymbol{\lambda}}$ and its inverse $f_{\boldsymbol{\lambda}}$ satisfy a system of Löwner-like equations

$$\begin{aligned} \frac{\partial f(z,\boldsymbol{\lambda})}{\partial\lambda_j} &= f(z,\boldsymbol{\lambda})\frac{\kappa_j(\boldsymbol{\lambda})+f(z,\boldsymbol{\lambda})}{\kappa_j(\boldsymbol{\lambda})-f(z,\boldsymbol{\lambda})}\frac{\partial\phi(\boldsymbol{\lambda})}{\partial\lambda_j},\\ \frac{\partial g(p,\boldsymbol{\lambda})}{\partial\lambda_j} &= p\frac{p+\kappa_j(\boldsymbol{\lambda})}{p-\kappa_j(\boldsymbol{\lambda})}\frac{\partial g(p,\boldsymbol{\lambda})}{\partial p}\frac{\partial\phi(\boldsymbol{\lambda})}{\partial\lambda_j}, \qquad j=1,\ldots,M, \end{aligned}$$

with respect to λ_j's. Secondly, one can choose the upper half plane H rather than the disc, and consider univalent conformal maps g_λ from H to a simply connected domain B_λ with ∞ on its boundary. The holomorphic functions $g(p,\boldsymbol{\lambda})$ and $f(z,\boldsymbol{\lambda})$ representing these maps satisfy a pair of differential equations called the 'chordal Löwner equation' (see below). This equation, too, has multi-slit analogues.

Remarkably, these Löwner-type equations also emerge in finite variable (or finite field) reductions of various dispersionless integrable systems. This fact was first discovered by Gibbons and Tsarev in the case of the Benney equations [2, 3]. $g(p,\boldsymbol{\lambda})$ shows up therein as a generating function $g(p)$ of the Benney moments, and the equations of motion of those moments are converted to an evolution equation for $g(p)$. This evolution equation can be identified with the Lax equation of the second flow of the dispersionless KP hierarchy. Gibbons and Tsarev observed that a general M-variable reduction of this system can be obtained in the following form:

(1) $g(p)$ depends on the spacetime variable (x,t) of the Benney equations via M reduced variables $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_M)$ as

$$g(p) = g(p,\boldsymbol{\lambda}), \qquad \boldsymbol{\lambda} = \boldsymbol{\lambda}(x,t).$$

(2) $g(p,\boldsymbol{\lambda})$ satisfies the chordal Löwner equations

$$\frac{\partial g(p,\boldsymbol{\lambda})}{\partial\lambda_j} = \frac{1}{p-U_j(\boldsymbol{\lambda})}\frac{\partial g(p,\boldsymbol{\lambda})}{\partial p}\frac{\partial a(\boldsymbol{\lambda})}{\partial\lambda_j}, \qquad j=1,\ldots,M,$$

of domains $B_{\boldsymbol{\lambda}}$ with M slits. The inverse function $f(z,\boldsymbol{\lambda})$ of $g(p,\boldsymbol{\lambda})$ satisfies the dual equations

$$\frac{\partial f(z,\boldsymbol{\lambda})}{\partial\lambda_j} = \frac{1}{U_j(\boldsymbol{\lambda})-f(z,\boldsymbol{\lambda})}\frac{\partial a(\boldsymbol{\lambda})}{\partial\lambda_j}, \qquad j=1,\ldots,M.$$

$U_j(\boldsymbol{\lambda})$ and $a(\boldsymbol{\lambda})$ are auxiliary functions that determine the reduction. $U_j(\boldsymbol{\lambda})$'s are called driving forces, and $a(\boldsymbol{\lambda})$ is related to the asymptotic behavior of $g(p,\boldsymbol{\lambda})$ and $f(z,\boldsymbol{\lambda})$ at infinity.

(3) λ_j's satisfy a system of 'diagonal' hydrodynamic equations

$$\frac{\partial \lambda_j}{\partial t} = \chi_j(\boldsymbol{\lambda}) \frac{\partial \lambda_j}{\partial x}, \qquad j = 1, \ldots, M.$$

Thus λ_j's are 'Riemann invariants' of the reduced system. The 'characteristic speeds' $\chi_j(\boldsymbol{\lambda})$ turn out to satisfy a set of conditions, which enables one to solve these equations by Tsarev's generalized hodograph method [4].

This result has been generalized to many other cases [5–11]. Recently this issue is also studied from the point of view of Hamiltonian structures [12].

In this paper, we address the problem of finite variable reductions of the universal Whitham hierarchy of genus zero [13]. This issue was studied by Guil *et al* [9], and the relevance of Löwner-type equations was already recognized therein. We reconsider this issue in a new perspective based on the dispersionless Hirota equations. As demonstrated in the case of the dispersionless KP and Toda hierarchies [11], the dispersionless Hirota equations can be a powerful tool for studying finite variable reductions of dispersionless integrable systems. The goal of this paper is to generalize this observation to the case of the universal Whitham hierarchy.

This paper is organized as follows. Section 2 is a brief review of the universal Whitham hierarchy of genus zero. Sections 3 and 4 deal with one-variable reduction. In section 3, we derive Löwner-type equations and diagonal hydrodynamic equations as necessary conditions for one-variable reduction. In section 4, we show that the hydrodynamic equations can be solved by the hodograph method, and lastly confirm that the Löwner-type equations and the hydrodynamic equations are sufficient to determine a solution of the universal Whitham hierarchy. Sections 5–7 are devoted to multi-variable reductions. In section 5, we examine 'rational reduction' (or 'algebraic orbits') of the universal Whitham hierarchy as a prototype of multi-variable reductions. Following the lines illustrated therein, we formulate a general form of multi-variable reduction in section 6, and reconsider the results of Guil *et al* [9] on the basis of this formulation. Section 8 is a summary of our results.

2. Universal Whitham hierarchy of genus zero

This section is a collection of basic notions and results on the universal Whitham hierarchy of genus zero picked out from the literature [13–16].

2.1. Lax equations

We consider the universal Whitham hierarchy of genus zero with $N + 1$ marked points on a Riemann sphere with coordinate p [13]. One of the marked points are fixed to $p = \infty$; the others $p = q_1, \ldots, q_N$ are part of the dynamical variables of the hierarchy. The other dynamical variables are the coefficients of the Laurent series

$$\begin{aligned} z_0(p) &= p + \sum_{j=2}^{\infty} u_{0j} p^{-j+1}, \\ z_\alpha(p) &= \frac{r_\alpha}{p - q_\alpha} + \sum_{j=1}^{\infty} u_{\alpha j} (p - q_\alpha)^{j-1} \qquad (\alpha = 1, \ldots, N), \end{aligned} \tag{1}$$

which are assumed to converge in domains $\mathcal{D}_0$ and $\mathcal{D}_\alpha$ (or asymptotic expansion of holomorphic functions at the boundary points $p = \infty$ and $p = q_\alpha$ of such domains). These Laurent series are dispersionless analogues of the Lax operators in dispersive integrable hierarchies.

In this setup, the universal Whitham hierarchy has $N + 1$ series of time variables $t_{\alpha n}$ ($n = 1, 2, \ldots$ for $\alpha = 0$ and $n = 0, 1, \ldots$ for $\alpha = 1, \ldots, N$). Let us use $\boldsymbol{t}$ to denote these time variables collectively, namely,

$$\boldsymbol{t} = (t_{01}, t_{02}, \ldots, t_{10}, t_{11}, \ldots, t_{N0}, t_{N1}, \ldots).$$

The lowest variables $t_{01}, t_{10}, \ldots, t_{N0}$ in each series $t_{\alpha n}, \alpha = 0, 1, \ldots, N$, play a special role. In particular, t_{01} should be interpreted as a *spatial* variable rather than a time variable. In a sense, $t_{10}, \ldots, t_{N0}$, too, are a kind of spatial variables, which originate in *charge* variables of an $N + 1$-component charged fermion system [16]. For convenience, we introduce the auxiliary variable

$$t_{00} = -\sum_{\alpha=1}^{N} t_{\alpha 0}.$$

(This stems from the constraint of total charge being zero.) We also use the abbreviation

$$\partial_{\alpha n} = \partial/\partial t_{\alpha n}.$$

for the derivatives in the time variables.

Time evolution of the system is defined by the dispersionless Lax equations [13]

$$\partial_{\alpha n} z_\beta(p) = \{\Omega_{\alpha n}(p), z_\beta(p)\} \qquad (\alpha, \beta = 0, 1, \ldots, N) \tag{2}$$

with respect to the two-dimensional Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial t_{01}} - \frac{\partial f}{\partial t_{01}}\frac{\partial g}{\partial p} \tag{3}$$

on the (p, t_{01}) space. Thus p is a conjugate variable of t_{01}, in other words, classical limit of the 'momentum' ∂_{01} in a one-dimensional quantum mechanical system. $\Omega_{0n}(p)$'s and $\Omega_{\alpha n}(p)$'s for $n \geqslant 1$ are polynomial in p and $(p - q_\alpha)^{-1}$,

$$\Omega_{0n}(p) = p^n + a_{0n2}p^{n-2} + \cdots + a_{0nn},$$
$$\Omega_{\alpha n}(p) = \frac{a_{\alpha n0}}{(p - q_\alpha)^n} + \frac{a_{\alpha n1}}{(p - q_\alpha)^{n-1}} + \cdots + \frac{a_{\alpha nn-1}}{(p - q_\alpha)},$$

and given by the singular part of Laurent expansion of $z_0(p)^n$ and $z_\alpha(p)^n$ (including the constant term for the former):

$$\begin{aligned} z_0(p)^n &= \Omega_{0n}(p) + O(p^{-1}) &\qquad (p \to \infty), \\ z_\alpha(p)^n &= \Omega_{\alpha n}(p) + O(1) &\qquad (p \to q_\alpha). \end{aligned} \tag{4}$$

The first few of them read

$$\Omega_{01}(p) = p, \qquad \Omega_{02}(p) = p^2 + 2u_{02}, \quad \ldots,$$
$$\Omega_{\alpha 1}(p) = \frac{r_\alpha}{p - q_\alpha}, \qquad \Omega_{\alpha 2}(p) = \frac{r_\alpha^2}{(p - q_\alpha)^2} + \frac{2r_\alpha u_{\alpha 1}}{p - q_\alpha}, \quad \ldots.$$

$\Omega_{\alpha 0}(p)$'s are exceptional and given by logarithmic functions:

$$\Omega_{\alpha 0}(p) = -\log(p - q_\alpha). \tag{5}$$

2.2. *S-functions and Hamilton–Jacobi equations*

We now introduce a set of new variables to extend the foregoing Lax equations to a larger system.

The first set of new variables are 'conjugate variables' of $z_\beta(p)$'s [14, 15], which play the role of the Orlov–Schulman operators in the present setup. As a consequence of the Lax equations, $\Omega_{\alpha n}(p)$'s satisfy the dispersionless Zakharov–Shabat equations [13]

$$\partial_{\beta n}\Omega_{\alpha m}(p) - \partial_{\alpha m}\Omega_{\beta n}(p) + \{\Omega_{\alpha m}(p), \Omega_{\beta n}(p)\} = 0 \qquad (\alpha, \beta = 0, 1, \ldots, N). \tag{6}$$

They can be packed into the equation

$$\omega \wedge \omega = 0 \tag{7}$$

for the closed 2-form

$$\omega = \sum_{n=1}^{\infty} \mathrm{d}\Omega_{0n}(p) \wedge \mathrm{d}t_{0n} + \sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} \mathrm{d}\Omega_{\alpha n}(p) \wedge \mathrm{d}t_{\alpha n},$$

where 'd' stands for exterior differentiation with respect to both $\boldsymbol{t}$ and p. By Darboux's theorem, ω can be expressed in a canonical form with two 'Darboux variables'. One can choose $z_\beta(p)$ as one of the Darboux variable; let $\zeta_\beta(p)$ denote the conjugate variable:

$$\omega = \mathrm{d}z_\beta(p) \wedge \mathrm{d}\zeta_\beta(p) \qquad (\beta = 0, 1, \ldots, N). \tag{8}$$

This equation implies that $\zeta_\beta(p)$'s satisfy Lax equations of the same form

$$\partial_{\alpha n}\zeta_\beta(p) = \{\Omega_{\alpha n}(p), \zeta_\beta(p)\} \qquad (\alpha, \beta = 0, 1, \ldots, N) \tag{9}$$

as $z_\beta(p)$'s, along with the canonical Poisson relations

$$\{z_\beta(p), \zeta_\beta(p)\} = 1 \qquad (\beta = 0, 1, \ldots, N). \tag{10}$$

Having introduced $\zeta_\beta(p)$'s, we can now introduce a second set of new variables $\mathcal{S}_\beta(p)$, $\beta = 0, 1, \ldots, N$, which are called 'S-functions' [9, 13–15]. To this end, we rewrite (8) as

$$\mathrm{d}(\theta + \zeta_\beta(p)\,\mathrm{d}z_\beta(p)) = 0,$$

where

$$\theta = \sum_{n=1}^{\infty} \Omega_{0n}(p)\,\mathrm{d}t_{0n} + \sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} \Omega_{\alpha n}\,\mathrm{d}t_{\alpha n}.$$

The S-function $\mathcal{S}_\beta(p)$ is defined as a potential of $\theta + \zeta_\beta(p)\,\mathrm{d}z_\beta(p)$:

$$\theta + \zeta_\beta(p)\,\mathrm{d}z_\beta(p) = \mathrm{d}\mathcal{S}_\beta(p). \tag{11}$$

Though one can find these S-functions as Laurent series of p (for $\beta = 0$) or of $(p - q_\alpha)^{-1}$ (for $\beta = 1, \ldots, N$), it is more convenient to consider them as Laurent series in $z_\beta(p)$. In such an expression, they can be expanded as

$$\begin{aligned} \mathcal{S}_0(p) &= \sum_{n=1}^{\infty} t_{0n} z_0(p)^n + t_{00}\log z_0(p) - \sum_{n=1}^{\infty} \frac{z_0(p)^{-n}}{n} v_{0n}, \\ \mathcal{S}_\beta(p) &= \sum_{n=1}^{\infty} t_{\beta n} z_\beta(p)^n + t_{\beta 0}\log z_\beta(p) + \phi_\beta - \sum_{n=1}^{\infty} \frac{z_\beta(p)^{-n}}{n} v_{\beta n}, \end{aligned} \tag{12}$$

where the coefficients v_{0n}, $v_{\beta n}$ and ϕ_β are functions of $\boldsymbol{t}$. We can rewrite this expression as

$$\mathcal{S}_0(p) = S_0(z_0(p)), \qquad \mathcal{S}_\beta(p) = S_\beta(z_\beta(p))$$

by introducing the functions

$$\begin{aligned} S_0(z) &= \sum_{n=1}^{\infty} t_{0n} z^n + t_{00}\log z - \sum_{n=1}^{\infty} \frac{z^{-n}}{n} v_{0n}, \\ S_\beta(z) &= \sum_{n=1}^{\infty} t_{\beta n} z^n + t_{\beta 0}\log z + \phi_\beta - \sum_{n=1}^{\infty} \frac{z^{-n}}{n} v_{\beta n} \end{aligned} \tag{13}$$

of z. These $S_\beta(z)$'s, too, are called 'S-functions'.

In a sense, the second set of S-functions $S_\beta(z)$ is more fundamental. They satisfy the Hamilton–Jacobi equations [9, 13]

$$\partial_{\alpha n} S_\beta(z) = \Omega_{\alpha n}(\partial_{01} S_\beta(z)) \qquad (\alpha = 0, 1, \ldots, N), \tag{14}$$

which are quasiclassical limit of the (scalar-valued) auxiliary linear equations of the underlying multi-component KP hierarchy [16]. Moreover, the t_{01}-derivative

$$p_\beta(z) = \partial_{01} S_\beta(z)$$

gives the inverse function of $z = z_\beta(p)$, namely,

$$z_\beta(p_\beta(z)) = z, \qquad p_\beta(z_\beta(p)) = p. \tag{15}$$

We can derive a simple relation among q_β, r_β and $\phi_\beta, v_{\beta 1}$ from these remarks. The foregoing Laurent expansion of the S-functions implies that $p_0(z)$ and $p_\beta(z)$ $(\beta = 1, \ldots, N)$ behave as

$$p_0(z) = z + O(z^{-1}), \qquad p_\beta(z) = \partial_{01}\phi_0 + \partial_{01} v_{\beta 1} z^{-1} + O(z^{-2})$$

as $z \to \infty$. Since the inverse function $p = p_\beta(z)$ of

$$z = z_\beta(p) = \frac{r_\beta}{p - q_\beta} + O(1)$$

should coincide with this expression of $p_\beta(z)$, we find that

$$q_\beta = -\partial_{01}\phi_\beta, \qquad r_\beta = -\partial_{01} v_{\beta 1}. \tag{16}$$

On the other hand, the Hamilton–Jacobi equations for $t_{\alpha 0}$ take such a form as

$$\partial_{\alpha 0} S_\beta(z) = -\log(\partial_{01} S_\beta(z) - q_\alpha) = -\log(p_\beta(z) - q_\alpha),$$

which can be solved for $p_\beta(z)$ as

$$p_\beta(z) = q_\alpha + \mathrm{e}^{-\partial_{\alpha 0} S_\beta(z)}.$$

Letting $\alpha = \beta$ and recalling the Laurent expansion of $S_\beta(z)$, we find that

$$p_\beta(z) = q_\beta + \mathrm{e}^{-\partial_{\alpha 0}\phi_0} z^{-1} + O(z^{-2}).$$

This gives another expression of r_β:

$$r_\beta = \mathrm{e}^{-\partial_{\alpha 0}\phi_\alpha}. \tag{17}$$

2.3. *F-function and Hirota equations*

We now introduce the F-function $\mathcal{F} = \mathcal{F}(\boldsymbol{t})$ (logarithm of the quasiclassical τ function [13]) as a solution of the equations

$$\begin{aligned} &\partial_{0n}\mathcal{F} = v_{0n}, \qquad \partial_{\alpha n}\mathcal{F} = v_{\alpha n}, \\ &\partial_{\alpha 0}\mathcal{F} = -\phi_\alpha + \sum_{\beta=1}^{\alpha} t_{\beta 0}\log(-1), \qquad \alpha = 1, \ldots, N, \end{aligned} \tag{18}$$

where $\log(-1)$ is understood to be equal to, say, $\pi\mathrm{i}$, though the choice of the branch is irrelevant in the final result. (This definition of the F-function [16] is slightly different from that of Mañas *et al* [14, 15], but this does not affect the main part of results.) This strange factor is related to the signature factors $\epsilon_{\alpha\beta}$ that we shall encounter below.

All relevant quantities of the hierarchy can be expressed in terms of the F-function. In particular, the S-functions have a very compact expression:

$$\begin{aligned} S_0(z) &= \sum_{n=1}^{\infty} t_{0n} z^n + t_{00} \log z - D_0(z)\mathcal{F}, \\ S_\alpha(z) &= \sum_{n=1}^{\infty} t_{\alpha n} z^n + t_{\alpha 0} \log z + \phi_\alpha - D_\alpha(z)\mathcal{F}, \end{aligned} \tag{19}$$

where

$$D_0(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{0n}, \qquad D_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \partial_{\alpha n}.$$

Consequently, the p-functions, too, can be neatly expressed as

$$p_0(z) = z - \partial_{01} D_0(z)\mathcal{F}, \qquad p_\alpha(z) = -\partial_{01}(D_\alpha(z) + \partial_{\alpha 0})\mathcal{F}. \tag{20}$$

The z-functions do not have such a simple expression, but one can determine the Laurent coefficients $u_{\alpha n}$ order by order. For example, let us derive an expression of $u_{02}, q_\alpha, r_\alpha$. The first few terms of the foregoing expression of the p-functions read

$$p_0(z) = z - \partial_{01}^2 \mathcal{F} z^{-1} + O(z^{-2}), \qquad p_\alpha(z) = -\partial_{01}\partial_{\alpha 0}\mathcal{F} - \partial_{01}\partial_{\alpha 1}\mathcal{F} z^{-1} + O(z^{-2}).$$

On the other hand, as the inverse function of

$$z_0(p) = p + u_{02} p^{-1} + O(p^{-2}), \qquad z_\alpha(p) = \frac{r_\alpha}{p - q_\alpha} + O(1),$$

$p_0(z)$ and $p_\alpha(z)$ should have Laurent expansion of the form

$$p_0(z) = z - u_{02} z^{-1} + O(z^{-2}), \qquad p_\alpha(z) = q_\alpha + r_\alpha z^{-1} + O(z^{-2}).$$

Thus we find that

$$u_{02} = \partial_{01}^2 \mathcal{F}, \qquad q_\alpha = -\partial_{01}\partial_{\alpha 0}\mathcal{F}, \qquad r_\alpha = -\partial_{01}\partial_{\alpha 1}\mathcal{F}. \tag{21}$$

Note that this expression of q_α and r_α can also be derived from (16).

When the universal Whitham hierarchy is viewed as dispersionless (or quasiclassical) limit of the (charged) multi-component KP hierarchy [16], the F-function is identified with the limit of logarithm of the tau function. One can thereby derive the 'dispersionless Hirota equations' (or, more appropriately, 'dispersionless differential Fay identities')

$$\begin{aligned} \mathrm{e}^{\hat{D}_0(z)\hat{D}_0(w)\mathcal{F}} &= 1 - \frac{\partial_{01}(\hat{D}_0(z) - \hat{D}_0(w))\mathcal{F}}{z - w}, \\ z\,\mathrm{e}^{\hat{D}_0(z)\hat{D}_\alpha(w)\mathcal{F}} &= z - \partial_{01}(\hat{D}_0(z) - \hat{D}_\alpha(w))\mathcal{F}, \\ \mathrm{e}^{\hat{D}_\alpha(z)\hat{D}_\alpha(w)\mathcal{F}} &= -\frac{zw\partial_{01}(\hat{D}_\alpha(z) - \hat{D}_\alpha(w))\mathcal{F}}{z - w}, \\ \epsilon_{\alpha\beta}\,\mathrm{e}^{\hat{D}_\alpha(z)\hat{D}_\beta(w)\mathcal{F}} &= -\partial_{01}(\hat{D}_\alpha(z) - \hat{D}_\beta(w))\mathcal{F} \qquad (\alpha \neq \beta) \end{aligned} \tag{22}$$

for $\alpha, \beta = 1, \ldots, N$, where

$$\hat{D}_\alpha(z) = \begin{cases} D_0(z) & (\alpha = 0), \\ D_\alpha(z) + \partial_{\alpha 0} & (\alpha \neq 0), \end{cases} \qquad \epsilon_{\alpha\beta} = \begin{cases} +1 & (\alpha \leqslant \beta), \\ -1 & (\alpha > \beta). \end{cases}$$

The signature factor $\epsilon_{\alpha\beta}$ stems from a fermionic representation of the tau functions of the multi-component KP hierarchy [17, 18].

These dispersionless Hirota equations are equivalent to the universal Whitham hierarchy itself [16]. This is a generalization of the results known for the dispersionless KP and Toda hierarchies [19–22]. The notion of Faber polynomials and Grunsky coefficients, which originate in complex analysis, play a crucial role here.

2.4. Faber polynomials and Grunsky coefficients

Faber polynomials and Grunsky coefficients are hidden in the foregoing setup of the universal Whitham hierarchy.

First of all, $\Omega_{0n}(z)$ and $\Omega_{\alpha n}(z)$ are nothing but the Faber polynomials of $p_0(z)$ and $p_\alpha(z)$, respectively [16]. Namely, these 'polynomials' (in p and in $(p - q_\alpha)^{-1}$, respectively) are characterized by the following generating functions:

$$\begin{aligned} \log \frac{p_0(z) - q}{z} &= -\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \Omega_{0n}(q), \\ \log \frac{q - p_\alpha(z)}{q - q_\alpha} &= -\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \Omega_{\alpha n}(q). \end{aligned} \tag{23}$$

Moreover, differentiating these generating functions by q yields the generating functions

$$\begin{aligned} \frac{1}{q - p_0(z)} &= -\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \Omega'_{0n}(q), \\ \frac{1}{q - p_\alpha(z)} - \frac{1}{q - q_\alpha} &= -\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \Omega'_{\alpha n}(q) \end{aligned} \tag{24}$$

for the derivatives

$$\Omega'_{0n}(q) = \frac{\partial \Omega_{0n}(q)}{\partial q}, \qquad \Omega'_{\alpha n}(q) = \frac{\partial \Omega_{\alpha n}(q)}{\partial q}.$$

It will be more suggestive (and convenient) to rewrite the second equation of (24) as

$$\frac{1}{q - p_\alpha(z)} = -\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \Omega'_{\alpha 0}(q) - \Omega'_{\alpha 0}(q). \tag{25}$$

These types of identities are called 'kernel formulae' in the literature [23, 24] (because the left-hand side may be thought of as a Cauchy kernel), and shown to be useful in some applications of dispersionless Hirota equations such as 'associativity equations' [20, 24].

Grunsky coefficients show up when we use (20) to rewrite the right-hand side of the dispersionless Hirota equations (22) as

$$\begin{aligned} &1 - \frac{\partial_{01}(\hat{D}_0(z) - \hat{D}_0(w))\mathcal{F}}{z - w} = p_0(z) - p_0(w), \\ &z - \partial_{01}(\hat{D}_0(z) - \hat{D}_\alpha(w))\mathcal{F} = p_0(z) - p_\alpha(w), \\ &-\frac{zw\partial_{01}(\hat{D}_\alpha(z) - \hat{D}_\alpha(w))\mathcal{F}}{z - w} = \frac{zw(p_\alpha(z) - p_\alpha(w))}{z - w}, \\ &-\partial_{01}(\hat{D}_\alpha(z) - \hat{D}_\beta(w))\mathcal{F} = p_\alpha(z) - p_\beta(w), \end{aligned}$$

and consider the logarithm of both hand sides of each equation. Equations (22) thus turn into the following equations:

$$\begin{aligned} \hat{D}_0(z)\hat{D}_0(w)\mathcal{F} &= \log \frac{p_0(z) - p_0(w)}{z - w}, \\ \hat{D}_0(z)\hat{D}_0(w)\mathcal{F} &= \log \frac{p_0(z) - p_\alpha(w)}{z}, \\ \hat{D}_\alpha(z)\hat{D}_\alpha(w)\mathcal{F} &= \log \frac{zw(p_\alpha(z) - p_\alpha(w))}{z - w}, \\ \hat{D}_\alpha(z)\hat{D}_\beta(w)\mathcal{F} &= \log \frac{p_\alpha(z) - p_\beta(w)}{\epsilon_{\alpha\beta}} \qquad (\alpha \neq \beta). \end{aligned} \tag{26}$$

As one can show using the definition of the Faber polynomials (23), these equations are, actually, a generating functional representation of the Hamilton–Jacobi equations (14); this fact lies in the heart of the aforementioned equivalence of the dispersionless Hirota equations and the universal Whitham hierarchy.

The right-hand side of (26) is nothing but generating functions of the (generalized) Grunsky coefficients $b_{\alpha m\beta n}$ of the p-functions:

$$\begin{aligned}
&\log\frac{p_0(z)-p_0(w)}{z-w} = -\sum_{m,n=1}^{\infty} z^{-m}w^{-n}b_{0m0n},\\
&\log\frac{p_0(z)-p_\alpha(w)}{z} = -\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} z^{-m}w^{-n}b_{0m\alpha n},\\
&\log\frac{zw(p_\alpha(z)-p_\alpha(w))}{z-w} = -\sum_{m,n=0}^{\infty} z^{-m}w^{-n}b_{\alpha m\alpha n},\\
&\log\frac{p_\alpha(z)-p_\beta(w)}{\epsilon_{\alpha\beta}} = -\sum_{m,n=0}^{\infty} z^{-m}w^{-n}b_{\alpha m\beta n} \qquad (\alpha\neq\beta).
\end{aligned} \tag{27}$$

In particular, the z^{-1} terms of the first and second equations yield the identities

$$w - p_0(w) = \sum_{n=1}^{\infty} w^{-n}b_{010n}, \qquad -p_\alpha(w) = -\sum_{n=0}^{\infty} w^{-n}b_{01\alpha n}. \tag{28}$$

One can thus recover the p-functions from the Grunsky coefficients.

Equations (26) show that all second derivatives of the F-function are given by the Grunsky coefficients as

$$\hat\partial_{\alpha m}\hat\partial_{\beta n}\mathcal{F} = -b_{\alpha m\beta n} \qquad (\alpha,\beta = 0,1,\ldots,N), \tag{29}$$

where we have introduced the rescaled derivatives

$$\hat\partial_{\alpha n} = \begin{cases} \frac{1}{n}\partial_{\alpha n} & (n\neq 0),\\ \partial_{\alpha 0} & (n=0).\end{cases}$$

Conversely, one can use (29) as defining equations of F. In that case, the Grunsky coefficients have to satisfy a set of integrability conditions:

$$\hat\partial_{\gamma n}b_{\alpha l\beta m} = \hat\partial_{\alpha l}b_{\gamma n\beta m}. \tag{30}$$

Once those integrability conditions are shown to be satisfied, one can obtain the F-function as a solution of these equations. Moreover, this simultaneously ensures that the dispersionless Hirota equations are also satisfied, because (29) are, after all, the dispersionless Hirota equations in disguise. This method for showing the existence of the F-function was first developed for the case of the dispersionless KP and Toda hierarchies [11]. We shall apply this method to reductions of the universal Whitham hierarchy.

3. Löwner-type equations in one-variable reduction

In the setup of one-variable reduction, the dynamical variables $u_{\alpha n}$ $(n = 0, 1, \ldots, u_{\alpha 0} = r_\alpha)$ and q_α are assumed to depend on $\boldsymbol{t}$ via a single function $\lambda = \lambda(\boldsymbol{t})$ as

$$u_{\alpha n} = u_{\alpha n}(\lambda(\boldsymbol{t})), \qquad q_\alpha = q_\alpha(\lambda(\boldsymbol{t})).$$

Consequently, $z_\alpha(p)$ and $p_\alpha(z)$ are thought of as functions of p and $\lambda(\boldsymbol{t})$:

$$z_\alpha(p) = z_\alpha(p, \lambda(\boldsymbol{t})), \qquad p_\alpha(z) = p_\alpha(z, \lambda(\boldsymbol{t})).$$

The goal of this section is to derive the following:

Theorem 1. *If $u_{\alpha n}$ and q_α are functions of a single variable $\lambda = \lambda(\boldsymbol{t})$, then there is a function $U = U(\lambda)$ of λ such that $p_\alpha(z) = p_\alpha(z, \lambda)$ and $z_\alpha(p) = z_\alpha(p, \lambda)$, $\alpha = 0, 1, \ldots, N$, satisfy the Löwner-type equations*

$$\frac{\partial p_\alpha(z)}{\partial \lambda} = \frac{1}{U - p_\alpha(z)} \frac{\partial u_{02}}{\partial \lambda}, \tag{31}$$

$$\frac{\partial z_\alpha(p)}{\partial \lambda} = \frac{z'_\alpha(p)}{p - U} \frac{\partial u_{02}}{\partial \lambda}, \tag{32}$$

where $z'_\alpha(p)$ denotes the p-derivative

$$z'_\alpha(p) = \frac{\partial z_\alpha(p)}{\partial p}.$$

Moreover, $\lambda = \lambda(\boldsymbol{t})$ satisfies the hydrodynamic equations

$$\partial_{\alpha n}\lambda = \chi_{\alpha n}(\lambda)\partial_{01}\lambda \tag{33}$$

with characteristic speeds

$$\chi_{\alpha n}(\lambda) = \Omega'_{\alpha n}(U).$$

This theorem is a generalization of the one-variable reduction of the KP and Toda hierarchies [11]. Let us explain some implications:

(1) In a quite general situation as the theorem assumes, we can say nothing about the shape of the domain of $p_\alpha(z)$ (equivalently, the range of $z_\alpha(p)$) and the position of U therein. In particular, it is *a priori* not evident whether the domain of $p_\alpha(z)$ is a slit domain like that assumed in the original setup of the Löwner equation. Since the reduction defined by (31) and (32) seems to be meaningful in such a general situation (cf various solutions of Löwner-type solutions known in the literature [5–11]), we do not restrict our consideration to slit domains.

(2) All equations of (31) and (32) have a common driving force U. In particular, (32) mean that $z_\alpha(p)$'s satisfy an identical linear differential equation of the form

$$\left(\frac{\partial}{\partial\lambda} - \frac{1}{p - U}\frac{\partial u_{02}}{\partial \lambda}\frac{\partial}{\partial p}\right) z_\alpha(p) = 0.$$

By the classical theory of characteristics, this implies that $z_\alpha(p)$'s are mutually functionally related. Namely, there are functions $f_\alpha(z)$ of one variable z such that

$$z_\alpha(p) = f_\alpha(z_0(p)), \qquad \alpha = 1, \ldots, N. \tag{34}$$

This agrees with the setup of reductions by Guil *et al* [9], though they assume the special form

$$f_\alpha(z) = \frac{1}{z - c_\alpha}, \tag{35}$$

where c_α's are constants.

(3) Equations (33) determine the time evolution of the reduced dynamical variable $\lambda = \lambda(\boldsymbol{t})$. The whole hierarchy is thus reduced to the hydrodynamic equations (33) with Riemann invariant λ and characteristic speeds $\chi_{\alpha n}(\lambda)$. These hydrodynamic equations can be solved by the hodograph method (see theorem 2).

(4) It is not obvious from the subsequent proof that the converse of the statement of the theorem also holds. Therefore one has to show, separately, that (31) (or (32)) and (33) lead to a solution of the universal Whitham hierarchy. We shall do it with the aid of the dispersionless Hirota equations (see theorem 3).

3.1. Proof of theorem 1: step 1

Differentiating both hand sides of the first equation of (26) by t_{01} yields the equation

$$\partial_{01} D_0(z) D_0(w)\mathcal{F} = \frac{\partial_{01} p_0(z) - \partial_{01} p_0(w)}{p_0(z) - p_0(w)}.$$

Note that $\hat{D}_0(z) = D_0(z)$. Let us rewrite this equation as

$$p_0(z) - p_0(w) = \frac{\partial_{01} p_0(z) - \partial_{01} p_0(w)}{\partial_{01} D_0(z) D_0(w)\mathcal{F}} = -\frac{\partial_{01} p_0(z)}{D_0(w) p_0(z)} + \frac{\partial_{01} p_0(w)}{D_0(z) p_0(w)}.$$

We have used (20) to rewrite the denominator $\partial_{01} D_0(z) D_0(w)\mathcal{F}$ as

$$\partial_{01} D_0(z) D_0(w)\mathcal{F} = -D_0(w) p_0(z) = -D_0(z) p_0(w).$$

By the chain rule, we can express the derivatives on the right-hand side as

$$\partial_{01} p_0(z) = \frac{\partial p_0(z)}{\partial\lambda}\partial_{01}\lambda, \qquad \partial_{01} p_0(w) = \frac{\partial p_0(w)}{\partial\lambda}\partial_{01}\lambda,$$

$$D_0(w) p_0(z) = \frac{\partial p_0(z)}{\partial\lambda} D_0(w)\lambda, \qquad D_0(z) p_0(w) = \frac{\partial p_0(w)}{\partial\lambda} D_0(z)\lambda.$$

Hence last equation reduces to

$$p_0(z) - p_0(w) = -\frac{\partial_{01}\lambda}{D_0(w)\lambda} + \frac{\partial_{01}\lambda}{D_0(z)\lambda}$$

or, equivalently,

$$p_0(z) - \frac{\partial_{01}\lambda}{D_0(z)\lambda} = p_0(w) - \frac{\partial_{01}\lambda}{D_0(w)\lambda}.$$

Therefore both hand sides of this equation are independent of z and w. Let $U_0 = U_0(\lambda)$ denote this quantity:

$$p_0(z) - \frac{\partial_{01}\lambda}{D_0(z)\lambda} = U_0. \tag{36}$$

On the other hand, applying $D_0(z)$ to both hand sides of the first formula of (21) and using (20) yield the identity

$$D_0(z) u_{02} = \partial_{01}^2 D_0(z)\mathcal{F} = -\partial_{01} p_0(z).$$

Again by the chain rule, the derivatives on both hand sides can be expressed as

$$D_0(z) u_{02} = \frac{\partial u_{02}}{\partial\lambda} D_0(z)\lambda, \qquad \partial_{01} p_0(z) = \frac{\partial p_0(z)}{\partial\lambda}\partial_{01}\lambda.$$

Thus we find that

$$\frac{\partial_{01}\lambda}{D_0(z)\lambda} = -\frac{\partial u_{02}/\partial\lambda}{\partial p_0(z)/\partial\lambda}. \tag{37}$$

We can use (37) to rewrite the foregoing equation (36) as

$$p_0(z) + \frac{\partial u_{02}/\partial\lambda}{\partial p_0(z)/\partial\lambda} = U_0.$$

This implies that $p_0(z)$ satisfies the Löwner-type equation

$$\frac{\partial p_0(z)}{\partial\lambda} = \frac{1}{U_0 - p_0(z)}\frac{\partial u_{02}}{\partial\lambda}. \tag{38}$$

3.2. Proof of theorem 1: step 2

We can repeat almost the same calculations for the second equation of (26).

Firstly, differentiating both hand sides by t_{01}, we obtain the equation

$$\partial_{01}\hat{D}_\alpha(z)\hat{D}_\alpha(w)\mathcal{F} = \frac{\partial_{01}p_\alpha(z) - \partial_{01}p_\alpha(w)}{p_\alpha(z) - p_\alpha(w)},$$

which can be rewritten as

$$p_\alpha(z) - p_\alpha(w) = \frac{\partial_{01}p_\alpha(z) - \partial_{01}p_\alpha(w)}{\partial_{01}\hat{D}_\alpha(z)\hat{D}_\alpha(w)\mathcal{F}} = -\frac{\partial_{01}p_\alpha(z)}{\hat{D}_\alpha(z)p_\alpha(z)} + \frac{\partial_{01}p_\alpha(w)}{\hat{D}_\alpha(w)p_\alpha(w)}.$$

By the chain rule, this equation reduces to

$$p_\alpha(z) - \frac{\partial_{01}\lambda}{\hat{D}_\alpha(z)\lambda} = p_\alpha(w) - \frac{\partial_{01}\lambda}{\hat{D}_\alpha(w)\lambda}.$$

This implies that both hand sides are actually independent of z and w. Thus we have the equation

$$p_\alpha(z) - \frac{\partial_{01}\lambda}{\hat{D}_\alpha(z)\lambda} = U_\alpha, \tag{39}$$

where $U_\alpha = U_\alpha(\lambda)$ is a function of λ only.

Secondly, we can derive, from the first formula of (21), the identity

$$\hat{D}_\alpha(z)u_{02} = \partial_{01}^2\hat{D}_\alpha(z)\mathcal{F} = -\partial_{01}p_\alpha(z).$$

By the chain rule, this identity reduces to

$$\frac{\partial_{01}\lambda}{\hat{D}_\alpha(z)} = -\frac{\partial u_{02}/\partial\lambda}{\partial p_\alpha(z)/\partial\lambda}. \tag{40}$$

Equations (39) and (40) imply that $p_\alpha(z)$ satisfies the Löwner-type equation

$$\frac{\partial p_\alpha(z)}{\partial\lambda} = \frac{1}{U_\alpha - p_\alpha(z)}\frac{\partial u_{02}}{\partial\lambda}. \tag{41}$$

3.3. Proof of theorem 1: step 3

Let us examine implications of the third and fourth equation of (26).

Differentiating the third equation by t_{01} yields the equation

$$p_0(z) - p_\alpha(w) = -\frac{\partial_{01}p_0(z)}{\hat{D}_\alpha(w)p_0(z)} + \frac{\partial_{01}p_\alpha(w)}{D_0(z)p_\alpha(w)}.$$

By the chain rule, this equation reduces to

$$p_0(z) - p_\alpha(w) = -\frac{\partial u_{02}/\partial\lambda}{\partial o_0(z)/\partial\lambda} + \frac{\partial u_{02}/\partial\lambda}{\partial p_\alpha(w)/\partial\lambda}.$$

By (38) and (41), we can rewrite the two terms on the right-hand side as

$$\frac{\partial u_{02}/\partial\lambda}{\partial p_0(z)/\partial\lambda} = U_0 - p_0(z), \qquad \frac{\partial u_{02}/\partial\lambda}{\partial p_\alpha(w)/\partial\lambda} = U_\alpha - p_\alpha(w).$$

Thus we find that

$$U_0 = U_\alpha, \qquad \alpha = 1, \ldots, N, \tag{42}$$

namely, (38) and (41) actually have an identical driving force U.

The same conclusion follows from the fourth equation of (26). Namely, differentiating this equation by t_{01} eventually leads to the identities $U_\alpha = U_\beta$.

3.4. Proof of theorem 1: step 4

We can derive the evolution equations (33) as follows.

Let us rewrite (37) as

$$D_0(z)\lambda = -\frac{\partial p_0(z)/\partial\lambda}{\partial u_{02}/\partial\lambda}\partial_{01}\lambda = -\frac{\partial_{01}\lambda}{U - p_0(z)}$$

and use the identity

$$\frac{1}{U - p_0(z)} = -\sum_{n=1}^{\infty} \frac{z^{-n}}{n}\Omega'_{0n}(U)$$

that can be obtained from the kernel formula (24) by letting $q = U$. The outcome is the equation

$$D_0(z)\lambda = \sum_{n=1}^{\infty} \frac{z^{-n}}{n}\Omega'_{0n}(U)\partial_{01}\lambda.$$

This is a generating functional form of the equations of (33) for $\alpha = 0, n = 1, 2, \ldots$.

In exactly the same way, using (40) and the kernel formula for $p_\alpha(z)$, we can derive the equations

$$\hat{D}_\alpha(z)\lambda = (D_\alpha(z) + \partial_{\alpha 0})\lambda = \left(\sum_{n=1}^{\infty} \frac{z^{-n}}{n}\Omega'_{\alpha n}(U) + \Omega'_{\alpha 0}(U)\right)\partial_{01}\lambda.$$

They are a generating functional form of the equations of (33) for $\alpha = 1, \ldots, N$ and $n = 0, 1, \ldots$.

4. Solutions from one-variable reduction

4.1. Hodograph method

The hydrodynamic equations (33) can be solved by the hodograph method. The hodograph method is extremely simplified in this case, because there is only one variable λ.

Theorem 2. *Let $F(\lambda)$ be an arbitrary function of λ, and $\lambda = \lambda(\boldsymbol{t})$ a function that satisfies the hodograph equation*

$$\sum_{n=1}^{\infty} t_{0n}\chi_{0n}(\lambda) + \sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} t_{\alpha n}\chi_{\alpha n}(\lambda) = F(\lambda). \tag{43}$$

Further assume that the regularity condition

$$\sum_{n=1}^{\infty} t_{0n}\frac{\partial\chi_{0n}(\lambda)}{\partial\lambda} + \sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} t_{\alpha n}\frac{\partial\chi_{\alpha n}(\lambda)}{\partial\lambda} \neq \frac{\partial F(\lambda)}{\partial\lambda} \tag{44}$$

holds for $\lambda = \lambda(\boldsymbol{t})$. Then $\lambda = \lambda(\boldsymbol{t})$ satisfies the hydrodynamic equations (33).

Proof. We differentiate both hand sides of the hodograph equation by $t_{\alpha n}$. By the chain rule, this yields the equations

$$\chi_{\alpha n}(\lambda) + \left(\sum_{m=1}^{\infty} t_{0m}\frac{\partial \chi_{0m}(\lambda)}{\partial\lambda} + \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}\frac{\partial \chi_{\beta m}(\lambda)}{\partial\lambda}\right)\partial_{\alpha n}\lambda = \frac{\partial F(\lambda)}{\partial\lambda}\partial_{\alpha n}\lambda,$$

hence

$$\chi_{\alpha n}(\lambda) = \left(\frac{\partial F(\lambda)}{\partial\lambda} - \sum_{m=1}^{\infty} t_{0m}\frac{\partial \chi_{0m}(\lambda)}{\partial\lambda} - \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}\frac{\partial \chi_{\beta m}(\lambda)}{\partial\lambda}\right)\partial_{\alpha n}\lambda.$$

In particular, letting $\alpha = 0$ and $n = 0$, we have the equation

$$1 = \left(\frac{\partial F(\lambda)}{\partial\lambda} - \sum_{m=1}^{\infty} t_{0m}\frac{\partial \chi_{0m}(\lambda)}{\partial\lambda} - \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}\frac{\partial \chi_{\beta m}(\lambda)}{\partial\lambda}\right)\partial_{01}\lambda.$$

If we multiply the last equation by $\chi_{\alpha n}(\lambda)$ and subtract it from the previous one, the outcome is the equation

$$0 = \left(\frac{\partial F(\lambda)}{\partial\lambda} - \sum_{m=1}^{\infty} t_{0m}\frac{\partial \chi_{0m}(\lambda)}{\partial\lambda} - \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}\frac{\partial \chi_{\beta m}(\lambda)}{\partial\lambda}\right)(\partial_{\alpha n}\lambda - \chi_{\alpha n}(\lambda)\partial_{01}\lambda).$$

By the regularity condition, we can drop the prefactor of $\partial_{\alpha n}\lambda - \chi_{\alpha n}(\lambda)\partial_{01}\lambda$ and obtain the hydrodynamic equations (33). □

4.2. Existence of F-function

In the last section, we derived the Löwner-type equations (31), (32) and the hydrodynamic equations (33), but we have not confirmed the converse, namely, whether these equations ensure that $u_{\alpha n} = u_\alpha(\lambda(\boldsymbol{t}))$ and $q_\alpha = q_\alpha(\lambda(\boldsymbol{t}))$ give a solution of the universal Whitham hierarchy.

We now prove that the converse is also true, following the idea presented in the end of section 2.4. This is also a generalization of the result in the case of the dispersionless KP and Toda hierarchies [11].

Theorem 3. *The integrability conditions (30) of (29) are satisfied in the foregoing setup of one-variable reduction. The F-function $\mathcal{F} = \mathcal{F}(\boldsymbol{t})$ thus defined by (29) gives a solution of the dispersionless Hirota equations (22).*

Proof. Let us first illustrate the calculations in the case where $\alpha = \beta = \gamma = 0$. We substitute $z = z_1$ and $w = z_2$ in the first generating function of (27), and apply $\hat{D}_0(z_3) = D_0(z_3)$ to both hand sides. This yields the generating function

$$\frac{D_0(z_3)(p_0(z_1) - p_0(z_2))}{p_0(z_1) - p_0(z_2)} = -\sum_{l,m,n=1}^{\infty} z_1^{-l}z_2^{-m}z_3^{-n}\hat{\partial}_{0n}b_{0l0m}$$

of $\hat{\partial}_{0n}b_{0l0m}$'s. On the other hand, recall that the hydrodynamic equations (33) have a generating functional representation (cf step 4 of the proof of theorem 1). In the case of $\alpha = 0$, it reads

$$D_0(z)\lambda = \frac{\partial_{01}\lambda}{p_0(z) - U}.$$

Using this equation, the Löwner-type equation for $p_0(z)$ and the chain rule, we can rewrite the left-hand side of the foregoing generating function of $\hat{\partial}_{0n} b_{0l0m}$'s as

$$\begin{aligned}\frac{D_0(z_3)(p_0(z_1)-p_0(z_2))}{p_0(z_1)-p_0(z_2)} &= \frac{D_0(z_3)\lambda}{p_0(z_1)-p_0(z_2)}\left(\frac{\partial p_0(z_1)}{\partial\lambda}-\frac{\partial p_0(z_2)}{\partial\lambda}\right)\\ &= \frac{\partial_{01}\lambda}{(p_0(z_3)-U)(p_0(z_1)-p_0(z_2))}\left(\frac{1}{U-p_0(z_1)}-\frac{1}{U-p_0(z_2)}\right)\frac{\partial u_{02}}{\partial\lambda}\\ &= -\frac{\partial_{01}\lambda}{(U-p_0(z_1))(U-p_0(z_2))(U-p_0(z_3))}\frac{\partial u_{02}}{\partial\lambda}.\end{aligned}$$

Since this quantity is symmetric in z_1 and z_3, we have the functional identity

$$\frac{D_0(z_3)(p_0(z_1)-p_0(z_2))}{p_0(z_1)-p_0(z_2)} = \frac{D_0(z_1)(p_0(z_3)-p_0(z_2))}{p_0(z_3)-p_0(z_2)}.$$

This implies the identities

$$\hat{\partial}_{0n} b_{0l0m} = \hat{\partial}_{0l} b_{0n0m}$$

of the coefficients, which are exactly the integrability conditions (30) for $\alpha=\beta=\gamma=0$.

In much the same way, staring with one of the generating functions of (27) and applying $\hat{D}_\gamma(z)$ to it, we obtain the generating functions

$$\frac{\hat{D}_\gamma(z_3)(p_\alpha(z_1)-p_\beta(z_2))}{p_\alpha(z_1)-p_\beta(z_2)} = -\sum_{l,m,n} z_1^{-l} z_2^{-m} z_3^{-n}\hat{\partial}_{\gamma n} b_{\alpha l\beta m}$$

of derivatives of the general Grunsky coefficients. Since the hydrodynamic equations (33) have the generating functional form

$$\hat{D}_\alpha(z)\lambda = \frac{\partial_{01}\lambda}{p_\alpha(z)-U},$$

we can rewrite the foregoing generating functions as

$$\frac{\hat{D}_\gamma(z_3)(p_\alpha(z_1)-p_\beta(z_2))}{p_\alpha(z_1)-p_\beta(z_2)} = -\frac{\partial_{01}\lambda}{(U-p_\alpha(z_1))(U-p_\beta(z_2))(U-p_\gamma(z_3))}\frac{\partial u_{02}}{\partial\lambda}.$$

This implies the functional identity

$$\frac{\hat{D}_\gamma(z_3)(p_\alpha(z_1)-p_\beta(z_2))}{p_\alpha(z_1)-p_\beta(z_2)} = \frac{\hat{D}_\alpha(z_1)(p_\gamma(z_3)-p_\beta(z_2))}{p_\gamma(z_3)-p_\beta(z_2)},$$

hence the integrability conditions (30) as expected. More precisely, we have to be careful about the difference of the four types of generating functions in (27), but this does not affect the final conclusion. □

Let us mention that this proof reveals an interesting feature of the integrability conditions (30). Namely, since the derivatives of the Grunsky coefficients are nothing but the third derivatives of the F-function, the functional identities in the proof imply that these third derivatives have a generating function of the form

$$\sum_{l,m,n} z_1^{-l} z_2^{-m} z_3^{-n}\hat{\partial}_{\alpha l}\hat{\partial}_{\beta m}\hat{\partial}_{\gamma n}\mathcal{F} = -\frac{\partial_{01}\lambda}{(U-p_\alpha(z_1))(U-p_\beta(z_2))(U-p_\gamma(z_3))}\frac{\partial u_{02}}{\partial\lambda}. \qquad (45)$$

This result is suggestive from the point of view of associativity equations [20, 24], because the third derivatives are fundamental quantities therein. We shall see that a similar result holds in multi-variable reductions as well.

5. Rational reductions (algebraic orbits)

Unfortunately, it seems difficult to extend the foregoing method for the one-variable reduction to multi-variable reductions. In the case of multi-variable reductions, we shall *start* from Löwner-type equations rather than *derive* them. The problem is how to find a correct form of Löwner-type equations. Though an answer to this question is presented in the work of Guil *et al* [9], we dare to take a different (heuristic) route that leads to the same answer.

Our strategy is to examine rational reductions of the universal Whitham hierarchy (or 'algebraic orbits' in the terminology of Krichever [13]) as a prototype of general multi-variable reductions. This class of reductions cover, for example, the Zakharov reduction of the Benney equations [25], reductions of the dispersionless KP hierarchy related to 2D topological field theories [26–28], a hydrodynamic reduction of the Boyer–Finley equation [29], etc.

The work of Ferapontov *et al* [29] is particularly suggestive, because their method is exactly based on Löwner-type equations and some related equations. They indeed used those equations to apply Tsarev's generalized hodograph method [4]. On the other hand, since they do not use Lax equations, one cannot readily see how to generalize their results to higher flows of an underlying hierarchy. Therefore it is essential to understand their method in the perspective of Lax equations.

Bearing these issues in mind, let us briefly look into rational reductions of the universal Whitham hierarchy. Let us mention that the following consideration is more or less parallel to the approach that Gibbons *et al* followed in the case of the Benney equations [2, 3, 5].

Setup of rational reduction. In a rational reduction, we assume that there is a rational function

$$E(p) = p^{k_0} + \sum_{n=2}^{k_0} a_{0n} p^{k_0-n} + \sum_{\alpha=1}^{N}\sum_{n=1}^{k_\alpha} \frac{a_{\alpha n}}{(p-q_\alpha)^n},$$

with poles at $p = \infty, q_1, \ldots, q_N$ such that $z_0(p)$ and $z_\alpha(p), \alpha = 1, \ldots, N$, are given by Laurent expansion of fractional powers of $E(p)$ as

$$z_0(p) = E(p)^{1/k_0} \qquad \text{(Laurent expansion at } p=\infty\text{)}$$
$$z_\alpha(p) = E(p)^{1/k_\alpha} \qquad \text{(Laurent expansion at } p=q_\alpha\text{)}.$$

Actually, $E(p)$ can have extra (dynamical) poles other than ∞ and q_α's. This is indeed the case for, e.g., the Zakharov reductions of the Benney equations and the hydrodynamic reduction of the Boyer–Finley equations. In those reductions, $E(p)$ takes such a form as

$$E(p) = p + \sum_{k=1}^{M-1} \frac{a_k}{p-b_k},$$

whereas the Benney equations and the Boyer–Finley equations are embedded into the one- and two-point universal Whitham hierarchies (in other words, the dispersionless KP and Toda hierarchy).

In this setup, the critical points

$$p = p_j, \qquad E'(p_j) = 0, \qquad j = 1, \ldots, M,$$

and the critical values

$$\lambda_j = E(p_j), \qquad j = 1, \ldots, M,$$

of $E(p)$ play the role of driving forces and Riemann invariants. More precisely, if $E(p)$ is sufficiently general, the critical values λ_j's can be used as full parameters (or 'moduli') of $E(p)$. Thus $E(p)$ is understood to be a function $E(p, \boldsymbol{\lambda})$ of p and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$.

Hydrodynamic equations for λ_j. The Lax equations (2) for the z-functions reduce to the Lax equations

$$\partial_{\alpha n}E(p) = \{\Omega_{\alpha n}, E(p)\} \qquad (\alpha = 0, 1, \ldots, N) \tag{46}$$

for $E(p)$. Written more explicitly, these equations read

$$\partial_{\alpha n}E(p) = \Omega'_{\alpha n}(p)\partial_{01}E(p) - E'(p)\partial_{01}\Omega_{\alpha n}(p).$$

Letting $p = p_j$ in this equation yields the equation

$$\partial_{\alpha n}E(p)|_{p=p_j} = \Omega'_{\alpha n}(p_j)\partial_{01}E(p)|_{p=p_j}.$$

On the other hand, by the chain rule, differentiating $\lambda_j = E(p_j)$ by $t_{\alpha n}$ gives the identity

$$\partial_{\alpha n}\lambda_j = E'(p_j)\partial_{\alpha n}p_j + \partial_{\alpha n}E(p)|_{p=p_j} = \partial_{\alpha n}E(p)|_{p=p_j}.$$

Thus we obtain the diagonal hydrodynamic equations

$$\partial_{\alpha n}\lambda_j = \chi_{\alpha n}(\boldsymbol{\lambda})\partial_{01}\lambda_j \qquad (\alpha = 0, 1, \ldots, N) \tag{47}$$

with characteristic speed

$$\chi_{\alpha n}(\boldsymbol{\lambda}) = \Omega'_{\alpha n}(p_j).$$

Note that p_j's are now understood to be algebraic functions $p_j = p_j(\boldsymbol{\lambda})$ defined by the equation $E'(p) = 0$. Thus (47) may be thought of as a closed evolutionary system for $\boldsymbol{\lambda}$. As we shall show in a more general case, these reduced equations can be solved by the generalized hodograph method.

Löwner-type equations for $E(p)$. We now consider $E(p)$ to be a function of p and $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{t})$, and use the chain rule to rewrite the Lax equations (46). Both hand sides the Lax equations can be thereby expressed as

$$\mathrm{LHS} = \sum_{j=1}^{M}\frac{\partial E(p)}{\partial\lambda_j}\partial_{\alpha n}\lambda_j = \sum_{j=1}^{M}\frac{\partial E(p)}{\partial\lambda_j}\Omega'_{\alpha n}(p_j)\partial_{01}\lambda_j$$

and

$$\mathrm{RHS} = \sum_{j=1}^{M}\left(\Omega'_{\alpha n}(p)\frac{\partial E(p)}{\partial\lambda_j} - E'(p)\frac{\partial\Omega_{\alpha n}(p)}{\partial\lambda_j}\right)\partial_{01}\lambda_j,$$

where we have used (47) as well. Thus the Lax equations reduce to

$$\sum_{j=1}^{M}\left((\Omega'_{\alpha n}(p) - \Omega'_{\alpha n}(p_j))\frac{\partial E(p)}{\partial\lambda_j} - E'(p)\frac{\partial\Omega_{\alpha n}(p)}{\partial\lambda_j}\right)\partial_{01}\lambda_j = 0.$$

Consequently, if $E(p) = E(p,\lambda)$ satisfies the equation

$$(\Omega'_{\alpha n}(p) - \Omega'_{\alpha n}(p_j))\frac{\partial E(p)}{\partial\lambda_j} - E'(p)\frac{\partial\Omega_{\alpha n}(p)}{\partial\lambda_j} = 0, \tag{48}$$

then for any solution $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{t})$ of (47), $E(p)|_{\boldsymbol{\lambda}=\boldsymbol{\lambda}(\boldsymbol{t})}$ gives a solution of the Lax equation.

Let us examine (48) in more detail. When $\alpha = 0$ and $n = 1$, this equation is a trivial identity. The lowest nontrivial one is the case where $\alpha = 0$ and $n = 2$. In this case, since $\Omega_{02}(p) = p^2 + 2u_{02}$, (48) reduces to

$$(p - p_j)\frac{\partial E(p)}{\partial\lambda_j} - E'(p)\frac{\partial u_{02}}{\partial\lambda_j} = 0$$

or, equivalently,

$$\frac{\partial E(p)}{\partial \lambda_j} = \frac{E'(p)}{p - p_j}\frac{\partial u_{02}}{\partial \lambda_j}. \tag{49}$$

This is exactly a Löwner-type equation. We can readily derive the equations

$$\frac{\partial z_\alpha(p)}{\partial \lambda_j} = \frac{z'_\alpha(p)}{p - p_j}\frac{\partial u_{02}}{\partial \lambda_j} \tag{50}$$

for the $z_\alpha(p)$'s, because the z-functions are given by fractional powers of $E(p)$. The p-functions, in turn, satisfy the dual equations

$$\frac{\partial p_\alpha(z)}{\partial \lambda_j} = \frac{1}{p_j - p_\alpha(z)}\frac{\partial u_{02}}{\partial \lambda_j}. \tag{51}$$

As regards the case for $n > 2$, (48) turn out to be redundant, namely, automatically satisfied if (49) is satisfied. This fact can be explained in a more general form; we shall return to this issue in section 6.2.

6. Multi-variable reductions

In view of the foregoing interpretation of rational reductions, it is now rather straightforward to find a correct formulation of general multi-variable reductions. In this section, we present this formulation and its implications. After all, this reduction procedure is nothing but the 'diagonal reduction' in the sense of Guil *et al* [9]. We, however, attempt to reformulate it along the lines that we have pursued in the case of one-variable reduction.

6.1. Löwner-type equations for $p_\alpha(z)$ and $z_\alpha(p)$

In an M-variable reduction, the fundamental dynamical variables $u_{\alpha n}$ ($n = 0, 1, \ldots, u_{\alpha 0} = r_\alpha$) and q_α are assumed to be functions of M-dimensional reduced dynamical variables $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$. The reduced dynamical variables, in turn, depend on $\boldsymbol{t}$ as $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{t})$ and eventually satisfy a set of diagonal hydrodynamic evolution equations.

Accordingly, the p-functions $p_\alpha(z)$ and the z-functions $z_\alpha(p), \alpha = 0, 1, \ldots, N$, are functions of p and $\boldsymbol{\lambda}$,

$$z_\alpha(p) = z_\alpha(p, \boldsymbol{\lambda}), \qquad p_\alpha(z) = p_\alpha(z, \boldsymbol{\lambda}).$$

We assume that these functions satisfy the Löwner-type equations

$$\frac{\partial p_\alpha(z)}{\partial \lambda_j} = \frac{1}{U_j - p_\alpha(z)}\frac{\partial u_{02}}{\partial \lambda_j}, \qquad j = 1, \ldots, M, \tag{52}$$

or the dual equations

$$\frac{\partial z_\alpha(p)}{\partial \lambda_j} = \frac{z'_\alpha(p)}{p - U_j}\frac{\partial u_{02}}{\partial \lambda_j}, \qquad j = 1, \ldots, M, \tag{53}$$

for a given set of driving forces $U_j = U_j(\boldsymbol{\lambda})$ and the auxiliary function $u_{02} = u_{02}(\boldsymbol{\lambda})$. Unlike the one-variable case, these auxiliary functions have to satisfy a set of integrability conditions. We shall present these conditions later on when we consider the hodograph method.

As in the case of one-variable reduction, the dual equations (53) imply that the z-functions are functionally related to each other by functions $f_\alpha(z)$ of one variable z as (34) shows. Though Guil *et al* [9] further assumed the special form (35), the following consideration is not limited to that case.

As a consequence of these Löwner-type equations, $z_\alpha(p)$'s turn out to have critical points at $p = U_j$, namely,

$$z'_\alpha(U_j) = 0. \tag{54}$$

This is an immediate consequence of the structure of (53). Since the left-hand side has no singularity at $p = U_j$, the pole of $1/(p - U_j)$ at $p = U_j$ has to be canceled by a zero of $z'_\alpha(p)$. This, however, does not imply that the critical values $z_\alpha(U_j)$ coincide with λ_j's. This is in accord with the general fact that the choice of Riemann invariants is not unique but allows large arbitrariness. A standard way will be to choose the critical values $z_0(U_j)$ of $z_0(z)$ as λ_j's; the critical values $z_\alpha(U_j)$ of the other z-functions are then functionally related to $z_0(U_j)$ as (34) implies.

6.2. Hydrodynamic equations for λ_j

In view of the results on one-variable and rational reductions, it seems plausible that the reduced variables $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{t})$ satisfy diagonal hydrodynamic equations of the form

$$\partial_{\alpha n}\lambda_j = \chi_{\alpha n j}(\boldsymbol{\lambda})\partial_{01}\lambda_j \tag{55}$$

with characteristic speeds

$$\chi_{\alpha n j}(\boldsymbol{\lambda}) = \Omega'_{\alpha n}(U_j).$$

The relation between the Lax equations (2) and these equations, however, is more delicate than in the case of rational reductions, because λ_j's in the present setup are *not* assumed to be given by the critical values of the z-functions. Therefore we cannot derive (55) by simply letting $p = U_j$ in the Lax equations. Nevertheless, we can confirm that (55) are correct equations to be satisfied by $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{t})$.

Theorem 4. *If (52) (or (53)) and (55) are satisfied, then the Lax equations (2) are also satisfied.*

Although this theorem can also be deduced from some other results that we shall show later on (theorem 6 and theorem 7) , we dare to present a direct proof here. The outline of the proof is parallel to the case of rational reductions. A clue is the following:

Lemma 1. *If $p_\alpha(z)$'s satisfy (52), then $\Omega_{\alpha n}(p)$'s satisfy the identities*

$$\frac{\partial\Omega_{\alpha n}(p)}{\partial\lambda_j} = \frac{\Omega'_{\alpha n}(p) - \Omega'_{\alpha n}(U_j)}{p - U_j}\frac{\partial u_{02}}{\partial\lambda_j}. \tag{56}$$

Proof. Let us first consider the case where $\alpha = 0$. We differentiate the generating function (23) of $\Omega_{0n}(p)$'s by λ_j and use (52). This yields the identity

$$\begin{aligned} -\sum_{n=1}^{\infty}\frac{z^{-n}}{n}\frac{\partial\Omega_{0n}(p)}{\partial\lambda_j} &= \frac{1}{p_0(z) - p}\frac{\partial p_0(z)}{\partial\lambda_j} \\ &= -\frac{1}{(p - p_0(z))(U_j - p_0(z))}\frac{\partial u_{02}}{\partial\lambda_j}. \end{aligned}$$

On the other hand, the kernel formula (24) implies another identity of the form

$$\begin{aligned} -\sum_{n=1}^{\infty}\frac{z^{-n}}{n}(\Omega'_{0n}(p) - \Omega'_{0n}(U_j)) &= \frac{1}{p - p_0(z)} - \frac{1}{U_j - p_0(z)} \\ &= \frac{U_j - p}{(p - p_0(z))(U_j - p_0(z))}. \end{aligned}$$

By comparing these identities, we find the identity

$$\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial \Omega_{0n}(p)}{\partial \lambda_j} = \sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\Omega'_{0n}(p) - \Omega'_{\alpha 0}(U_j)}{p - U_j},$$

which implies that (56) holds for $\alpha = 0$. In much the same way using the second kernel formula (25), we can derive (56) for $\alpha = 1, \ldots, N$. □

Proof of theorem 4. The reasoning in the case of rational reductions also works in this case with slightest modification. Firstly, using (55) and the chain rule, we can rewrite the Lax equations (2) as

$$\sum_{j=1}^{M} \left((\Omega'_{\alpha n}(p) - \Omega'_{\alpha n}(U_j)) \frac{\partial z_\beta(p)}{\partial \lambda_j} - z'_\beta(p) \frac{\partial \Omega_{\alpha n}(p)}{\partial \lambda_j} \right) \partial_{01}\lambda_j = 0.$$

On the other hand, combining equations (53) for $z_\beta(p)$ with the identity (56) of the lemma yields the equations

$$\frac{\partial z_\beta(p)}{\partial \lambda_j} = \frac{z'_\beta(p)}{\Omega'_{\alpha n}(p) - \Omega'_{\alpha n}(U_j)} \frac{\partial \Omega_{\alpha n}(p)}{\partial \lambda_j}, \tag{57}$$

which implies that the last equations are indeed satisfied. □

Let us note that the last equations (57) amount to (48) in the previous section. As mentioned therein, (48) contains the Löwner-type equations (49) as a special case with $\alpha = 0$ and $n = 2$. In the present setup, (53) is a special case of (57) with $\alpha = 0$ and $n = 2$. By the way, lemma 1 (or its proof) says that (57) is a consequence of (53). This explains why (48) are 'redundant' in the setup of the last section.

6.3. Hodograph method

The hydrodynamic equations (55) can be solved by Tsarev's hodograph method [4]. As it turns out below, this is a straightforward generalization of the framework developed by Gibbons and Tsarev [2, 3] for reductions of the Benney equations.

In the multi-variable case, (52) and (53) have to satisfy a set of integrability conditions. As regards (52), the integrability conditions can be derived by eliminating the $\boldsymbol{\lambda}$-derivatives of $p_\alpha(z)$ from the identity

$$\frac{\partial}{\partial \lambda_j} \left(\frac{1}{U_k - p_\alpha(z)} \frac{\partial u_{02}}{\partial \lambda_k} \right) = \frac{\partial}{\partial \lambda_k} \left(\frac{1}{U_j - p_\alpha(z)} \frac{\partial u_{02}}{\partial \lambda_j} \right).$$

After some algebra, these conditions reduce to the equations

$$\begin{aligned} \frac{\partial U_k}{\partial \lambda_j} &= \frac{1}{U_j - U_k} \frac{\partial u_{02}}{\partial \lambda_j}, \\ \frac{\partial^2 u_{02}}{\partial \lambda_j \partial \lambda_k} &= \frac{2}{(U_j - U_k)^2} \frac{\partial u_{02}}{\partial \lambda_j} \frac{\partial u_{02}}{\partial \lambda_k}, \end{aligned} \tag{58}$$

which take exactly the same form as the equations derived by Gibbons and Tsarev in the case of the Benney equations. The same equations can be derived from the integrability conditions

$$\left[\frac{\partial}{\partial \lambda_j} - \frac{1}{p - U_j} \frac{\partial u_{02}}{\partial \lambda_j} \frac{\partial}{\partial p}, \frac{\partial}{\partial \lambda_k} - \frac{1}{p - U_k} \frac{\partial u_{02}}{\partial \lambda_k} \frac{\partial}{\partial p} \right] = 0 \tag{59}$$

of the dual equations (53) as well.

An important consequence of these equations is the following:

Lemma 2. *The characteristic speeds* $\chi_{\alpha nj} = \chi_{\alpha nj}(\boldsymbol{\lambda})$ *of (55) satisfy the equations*

$$\frac{\partial \chi_{\alpha nk}}{\partial \lambda_j} = (\chi_{\alpha nj} - \chi_{\alpha nk}) V_{jk}, \tag{60}$$

where

$$V_{jk} = \frac{1}{(U_j - U_k)^2} \frac{\partial u_{02}}{\partial \lambda_j}. \tag{61}$$

Proof. Let us first consider the case where $\alpha = 0$. Substituting $q = U_k$ in the kernel formula (24), yields the identity

$$-\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \chi_{0nk} = \frac{1}{U_k - p_0(z)}.$$

We now differentiate both hand sides by λ_j. Using (52) and (58), we can rewrite the outcome as

$$\begin{aligned}
-\sum_{n=1}^{\infty} \frac{z^{-n}}{n} \frac{\partial \chi_{0nk}}{\partial \lambda_j} &= -\frac{1}{(U_k - p_0(z))^2} \left(\frac{\partial U_k}{\partial \lambda_j} - \frac{\partial p_0(z)}{\partial \lambda_j} \right) \\
&= -\frac{1}{(U_k - p_0(z))^2} \left(\frac{1}{U_j - U_k} - \frac{1}{U_j - p_0(z)} \right) \frac{\partial u_{02}}{\partial \lambda_j} \\
&= -\frac{1}{(U_k - p_0(z))(U_j - p_0(z))(U_j - U_k)} \frac{\partial u_{02}}{\partial \lambda_j} \\
&= \left(\frac{1}{U_j - p_0(z)} - \frac{1}{U_k - p_0(z)} \right) \frac{1}{(U_j - U_k)^2} \frac{\partial u_{02}}{\partial \lambda_j} \\
&= -\sum_{n=1}^{\infty} \frac{z^{-n}}{n} (\chi_{\alpha nj} - \chi_{\alpha nk}) \frac{1}{(U_j - U_k)^2} \frac{\partial u_{02}}{\partial \lambda_j}.
\end{aligned}$$

This shows that (60) are indeed satisfied for $\alpha = 0$. We can confirm (60) for $\alpha = 1, \ldots, N$ in the same way, now using the second kernel formula (25). □

We can now formulate the hodograph method for (55) as follows.

Theorem 5. *Let* $F_j = F_j(\boldsymbol{\lambda})$, $j = 1, \ldots, M$, *be a set of functions of* $\boldsymbol{\lambda}$ *that satisfy the equations*

$$\frac{\partial F_k}{\partial \lambda_j} = (F_j - F_k) V_{jk}, \tag{62}$$

and $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{t})$ *a solution of the hodograph equations*

$$\sum_{n=1}^{\infty} t_{0n} \chi_{0nj}(\boldsymbol{\lambda}) + \sum_{\alpha=1}^{N} \sum_{n=0}^{\infty} t_{\alpha n} \chi_{\alpha nj}(\boldsymbol{\lambda}) = F_j(\boldsymbol{\lambda}), \qquad j = 1, \ldots, N. \tag{63}$$

Further assume that the regularity conditions

$$\sum_{n=1}^{\infty} t_{0n} \frac{\partial \chi_{0nj}(\boldsymbol{\lambda})}{\partial \lambda_j} + \sum_{\alpha=1}^{N} \sum_{n=0}^{\infty} t_{\alpha n} \frac{\partial \chi_{\alpha nj}(\boldsymbol{\lambda})}{\partial \lambda_j} \neq \frac{\partial F_j(\boldsymbol{\lambda})}{\partial \lambda_j}, \qquad j = 1, \ldots, N, \tag{64}$$

hold for $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{t})$. *Then* $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{t})$ *satisfies the hydrodynamic equations (55).*

Proof. We differentiate both hand sides of (63) by $t_{\alpha n}$. By the chain rule, this yields the equations

$$\chi_{\alpha nj} + \sum_{k=1}^{M}\left(\sum_{m=1}^{\infty} t_{0m}\frac{\partial\chi_{0mj}}{\partial\lambda_k} + \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}\frac{\partial\chi_{\beta mj}}{\partial\lambda_k}\right)\partial_{\alpha n}\lambda_k = \sum_{k=1}^{M}\frac{\partial F_j}{\partial\lambda_k}\partial_{\alpha n}\lambda_k,$$

hence

$$\chi_{\alpha nj} = \sum_{k=1}^{M}\left(\frac{\partial F_j}{\partial\lambda_k} - \sum_{m=1}^{\infty} t_{0m}\frac{\partial\chi_{0mj}}{\partial\lambda_k} - \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}\frac{\partial\chi_{\beta mj}}{\partial\lambda_k}\right)\partial_{\alpha n}\lambda_k.$$

Let us examine the quantity inside the parenthesis on the right-hand side. If $k \neq j$, we can use (60) and (62) to rewrite this quantity as

$$\begin{aligned}
&\frac{\partial F_j}{\partial\lambda_k} - \sum_{m=1}^{\infty} t_{0m}\frac{\partial\chi_{0mj}}{\partial\lambda_k} - \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}\frac{\partial\chi_{\beta mj}}{\partial\lambda_k}\\
&\quad = (F_k - F_j)V_{kj} - \sum_{m=1}^{\infty} t_{0m}(\chi_{0mk} - \chi_{0mj})V_{kj} - \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}(\chi_{\beta mk} - \chi_{\beta mj})V_{kj}\\
&\quad = \left(F_k - \sum_{m=1}^{\infty} t_{0m}\chi_{0mk} - \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}\chi_{\beta mk} - (k \text{ replaced with } j)\right)V_{kj}
\end{aligned}$$

which vanishes by (63). Thus the last equation simplifies as

$$\chi_{\alpha nj} = \left(\frac{\partial F_j}{\partial\lambda_j} - \sum_{m=1}^{\infty} t_{0m}\frac{\partial\chi_{0mj}}{\partial\lambda_j} - \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}\frac{\partial\chi_{\beta mj}}{\partial\lambda_j}\right)\partial_{\alpha n}\lambda_j.$$

Moreover, if $\alpha = 0$ and $n = 2$, this equation reduces to

$$1 = \left(\frac{\partial F_j}{\partial\lambda_j} - \sum_{m=1}^{\infty} t_{0m}\frac{\partial\chi_{0mj}}{\partial\lambda_j} - \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}\frac{\partial\chi_{\beta mj}}{\partial\lambda_j}\right)\partial_{01}\lambda_j.$$

If we multiply the last equation by $\chi_{\alpha nj}$ and subtract it from the previous one, the outcome is the equation

$$0 = \left(\frac{\partial F_j}{\partial\lambda_j} - \sum_{m=1}^{\infty} t_{0m}\frac{\partial\chi_{0mj}}{\partial\lambda_j} - \sum_{\beta=1}^{N}\sum_{m=0}^{\infty} t_{\beta m}\frac{\partial\chi_{\beta mj}}{\partial\lambda_j}\right)(\partial_{\alpha n}\lambda_j - \chi_{\alpha nj}\partial_{01}\lambda_j).$$

By the regularity condition, we can drop the prefactor of $\chi_{\alpha n}\lambda_j - \chi_{\alpha nj}\partial_{01}\lambda_j$ and obtain the hydrodynamic equations (55). □

We are thus eventually left with the problem of finding F_j's that satisfy (62). Such functions are given by contour integrals of the form

$$F_j = \sum_{\alpha=0}^{N}\oint_{C_\alpha}\frac{\mathrm{d}p}{2\pi\mathrm{i}}\frac{G_\alpha(z_\alpha(p))}{(p - U_j)^2}, \qquad (65)$$

where $G_\alpha(z)$ is an arbitrary holomorphic function of one variable z defined on the range of the map $p \mapsto z_\alpha(p)$, and C_α is a closed curve (or cycle) in the domain of $z_\alpha(p)$. It is not difficult to show that these F_j's satisfy (62) as a consequence of (53). Note that the characteristic speeds $\chi_{\alpha nj}$ themselves have such a contour integral representation as

$$\chi_{\alpha nj} = \oint \frac{\mathrm{d}p}{2\pi\mathrm{i}} \frac{z_\alpha(p)^n}{(p - U_j)^2}, \tag{66}$$

where the path of integral is a small circle encircling the point $p = \infty$ $(\alpha = 0)$ or $p = q_\alpha$ $(\alpha = 1, \ldots, N)$.

6.4. Existence of F-function

Theorem 3 and its proof can be generalized to the present setup without substantial modifications.

Theorem 6. *The integrability conditions (30) of (29) are satisfied in the foregoing setup of multi-variable reduction. The F-function $\mathcal{F} = \mathcal{F}(\boldsymbol{t})$ thus defined by (29) gives a solution of the dispersionless Hirota equations (22).*

Proof. We can proceed just as in the proof of theorem 3. By applying $\hat{D}_\gamma(z)$ to the generating functions (27), we obtain the generating functions

$$\frac{\hat{D}_\gamma(z_3)(p_\alpha(z_1) - p_\beta(z_2))}{p_\alpha(z_1) - p_\beta(z_2)} = -\sum_{l,m,n} z_1^{-l} z_2^{-m} z_3^{-n} \hat{\partial}_{\gamma n} b_{\alpha l \beta m}$$

of derivatives of the Grunsky coefficients. By the kernel formulae (24) and (25), the hydrodynamic equations (55) can be cast into the generating functional form

$$\hat{D}_\alpha(z)\lambda_j = \frac{\partial_{01}\lambda_j}{p_\alpha(z) - U_j}.$$

We can thereby rewrite the foregoing generating function as

$$\frac{\hat{D}_\gamma(z_3)(p_\alpha(z_1) - p_\beta(z_2))}{p_\alpha(z_1) - p_\beta(z_2)} = -\sum_{j=1}^{M} \frac{\partial_{01}\lambda_j}{(U_j - p_\alpha(z_1))(U_j - p_\beta(z_2))(U_j - p_\gamma(z_3))} \frac{\partial u_{02}}{\partial \lambda_j}.$$

This implies the functional identity

$$\frac{\hat{D}_\gamma(z_3)(p_\alpha(z_1) - p_\beta(z_2))}{p_\alpha(z_1) - p_\beta(z_2)} = \frac{\hat{D}_\alpha(z_1)(p_\gamma(z_3) - p_\beta(z_2))}{p_\gamma(z_3) - p_\beta(z_2)}.$$

The integrability conditions (30) follow from this identity immediately. The rest of the statement of the theorem is a consequence of the comments in the end of section 2.4. □

As a byproduct of this proof, we obtain the following generalization of (45) to multi-variable reductions:

$$\sum_{l,m,n} z_1^{-l} z_2^{-m} z_3^{-n} \hat{\partial}_{\alpha l} \hat{\partial}_{\beta m} \hat{\partial}_{\gamma n} \mathcal{F} = -\sum_{j=1}^{M} \frac{\partial_{01}\lambda_j}{(U_j - p_\alpha(z_1))(U_j - p_\beta(z_2))(U_j - p_\gamma(z_3))} \frac{\partial u_{02}}{\partial \lambda_j}. \tag{67}$$

7. *S*-functions in multi-variable reductions

As an application of the foregoing formulation of multi-variable reductions, we now reconsider the construction of *S*-functions by Guil *et al* [9]. As it turns out below, some part of their construction can be made more transparent with the aid of the identities (56) of lemma 1.

The construction of S-functions by Guil *et al* is based on hodograph solutions of the hydrodynamic equations (55). Given such a solution along with the p-functions satisfying (52), they construct the S-function in such a form as

$$\mathcal{S}_\beta(p) = \sum_{n=1}^{\infty} t_{0n}\Omega_{0n}(p) + \sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} t_{\alpha n}\Omega_{\alpha n}(p) + \mathcal{S}_{\beta-}(p) \tag{68}$$

or, equivalently,

$$S_\beta(z) = \sum_{n=1}^{\infty} t_{0n}\Omega_{0n}(p_\beta(z)) + \sum_{\alpha=1}^{N}\sum_{n=0}^{\infty} t_{\alpha n}\Omega_{\alpha n}(p_\beta(z)) + \mathcal{S}_{\beta-}(p_\beta(z)), \tag{69}$$

where $\mathcal{S}_{\beta-}(p)$ are required to satisfy the equations

$$\left(\frac{\partial}{\partial\lambda_j} - \frac{1}{p-U_j}\frac{\partial u_{02}}{\partial\lambda_j}\frac{\partial}{\partial p}\right)\mathcal{S}_{\beta-}(p) = -\frac{F_j}{p-U_j}\frac{\partial u_{02}}{\partial\lambda_j}. \tag{70}$$

Since the differential operators on the left-hand side are commutative, see (59), the integrability conditions of these *inhomogeneous* Löwner equations are given by

$$\left(\frac{\partial}{\partial\lambda_j} - \frac{1}{p-U_j}\frac{\partial u_{02}}{\partial\lambda_j}\frac{\partial}{\partial p}\right)\left(\frac{F_k}{p-U_k}\frac{\partial u_{02}}{\partial\lambda_k}\right) = \left(\frac{\partial}{\partial\lambda_k} - \frac{1}{p-U_k}\frac{\partial u_{02}}{\partial\lambda_k}\frac{\partial}{\partial p}\right)\left(\frac{F_j}{p-U_j}\frac{\partial u_{02}}{\partial\lambda_j}\right).$$

By straightforward calculations, one can see that these conditions are equivalent to equations (62) for F_j's. Thus the existence of a solution to (70) is ensured in the setup of the hodograph solution. With these definitions, the main part of the results of Guil *et al* can be stated as follows:

Theorem 7 (Guil *et al* [9])**.** *The S-functions $S_\beta(z)$, $\beta = 0, 1, \ldots, N$, satisfy the Hamilton–Jacobi equations (14).*

Let us prove this theorem using our tools. Firstly, by the chain rule, the $t_{\alpha n}$-derivative of $S_\beta(z)$ can be expanded as

$$\begin{aligned}\partial_{\alpha n} S_\beta(z) = \Omega_{\alpha n}(p_\beta(z)) &+ \sum_{j=1}^{M}\sum_{\gamma,m} t_{\gamma m}\left(\Omega'_{\gamma m}(p_\beta(z))\frac{\partial p_\beta(z)}{\partial\lambda_j} + \left.\frac{\partial\Omega_{\gamma m}(p)}{\partial\lambda_j}\right|_{p=p_\beta(z)}\right)\partial_{\alpha n}\lambda_j \\ &+ \sum_{j=1}^{M}\left(\mathcal{S}'_{\beta-}(p_\beta(z))\frac{\partial p_\beta(z)}{\partial\lambda_j} + \left.\frac{\partial\mathcal{S}_{\beta-}(p)}{\partial\lambda_j}\right|_{p=p_\beta(z)}\right)\partial_{\alpha n}\lambda_j,\end{aligned} \tag{71}$$

where we have used the abbreviated notation

$$\sum_{\gamma m} A_{\gamma m} = \sum_{m=1}^{\infty} A_{0m} + \sum_{\gamma=1}^{N}\sum_{m=0}^{\infty} A_{\gamma m}.$$

Using the Löwner-like equations (52) and the identity (56) of lemma 1, we can calculate the quantity in the first parenthesis on the right-hand side of (71) as

$$\begin{aligned}&\Omega'_{\gamma m}(p_\beta(z))\frac{\partial p_\beta(z)}{\partial\lambda_j} + \left.\frac{\partial\Omega_{\gamma m}(p)}{\partial\lambda_j}\right|_{p=p_\beta(z)} \\ &\quad= \frac{\Omega'_{\gamma m}(p_\beta(z))}{U-p_\beta(z)}\frac{\partial u_{02}}{\partial\lambda_j} + \frac{\Omega'_{\gamma m}(p_\beta(z)) - \Omega'_{\gamma m}(U_j)}{p_\beta(z)-U_j}\frac{\partial u_{02}}{\partial\lambda_j} \\ &\quad= -\frac{\Omega'_{\gamma m}(U_j)}{p_\beta(z)-U_j}\frac{\partial u_{02}}{\partial\lambda_j}.\end{aligned}$$

As regards the quantity in the second parenthesis, we use equations (70) satisfied by $\mathcal{S}_{\beta-}(z)$ as

$$\mathcal{S}'_{\beta-}(p_\beta(z))\frac{\partial p_\beta(z)}{\partial\lambda_j}+\frac{\partial\mathcal{S}_{\beta-}(p)}{\partial\lambda_j}\bigg|_{p=p_\beta(z)}=\frac{\mathcal{S}'_{\beta-}(p_\beta(z))}{U_j-p_\beta(z)}\frac{\partial u_{02}}{\partial\lambda_j}+\frac{\partial\mathcal{S}_{\beta-}(p)}{\partial\lambda_j}\bigg|_{p=p_\beta(z)}$$
$$=\left(\frac{1}{U_j-p}\frac{\partial u_{02}}{\partial\lambda_j}\frac{\partial\mathcal{S}_{\beta-}(p)}{\partial p}+\frac{\partial\mathcal{S}_{\beta-}(p)}{\partial\lambda_j}\right)\bigg|_{p=p_\beta(z)}$$
$$=-\frac{F_j}{p_\beta(z)-U_j}\frac{\partial u_{02}}{\partial\lambda_j}.$$

Consequently, (71) turns into an equation of the form

$$\partial_{\alpha n}S_\beta(z)=\Omega_{\alpha n}(p_\beta(z))+\sum_{j=1}^{M}\left(\sum_{\gamma m}t_{\gamma m}\Omega'_{\gamma m}(U_j)+F_j\right)\frac{\partial_{\alpha n}\lambda_j}{p_\beta(z)-U_j}\frac{\partial u_{02}}{\partial\lambda_j}.$$

Since each term of the sum on the right-hand side vanishes by the hodograph equations (63), we have the equation

$$\partial_{\alpha n}S_\beta(z)=\Omega_{\alpha n}(p_\beta(z)).$$

In the case where $\alpha=0$ and $n=1$, this equation reduces to

$$\partial_{01}S_\beta(z)=p_\beta(z),$$

by which we can eliminate $p_\beta(z)$ from the last equation and obtain the Hamilton–Jacobi equations (14).

This completes the proof of the theorem. Note that using the identities (56) makes the proof shorter and more understandable than the original proof of Guil *et al*.

8. Conclusion

We have thus seen that Löwner-type equations play a fundamental role in finite variable reductions of the universal Whitham hierarchy of genus zero. The status of dispersionless Hirota equations therein is more subtle. As regards the one-variable reduction, the dispersionless Hirota equations (22) are certainly a clue. The generating functional form (26) of these equations enabled us to derive the Löwner-type equations (31) and (32) directly from the assumption that all dynamical variables are functions of a single reduced variable λ. Unfortunately, this method does not work for multi-variable reductions. Therefore we were forced to start from (rather than derive) the Löwner-type equations (52) and (53) and to confirm that they are a correct set of reduction conditions (theorem 4). For both one-variable and the multi-variable reductions, however, we could eventually justify the reduction procedure in a unified way, namely, by proving that the defining equations (29) of the F-function is integrable (theorems 3 and 6). This is a place where the dispersionless Hirota equations play a truly fundamental role.

Viewed from a technical point of view, another clue of our method is the use of the generating functions (23) and (27) and various identities derived therefrom. Not only being closely related to the dispersionless Hirota equations themselves, these generating functions turned out to be also extremely useful in many aspects of finite variable reductions.

Our next target will be, naturally, the cases of nonzero genera [13]. It will be rather easy to derive dispersionless Hirota equations for those cases, as partly argued by Krichever *et al* in a different setup [30]. A main problem is to find a correct form of Löwner-type equations. We expect to find a prototype in differential geometry of Hurwitz spaces [31, 32] and associated Whitham-type hierarchies [13, 27], because they are generalizations of the rational reductions that we considered as a prototype of general multi-variable reductions for the genus zero case. Presumably, we should start with the genus one case, for which an explicit description of Hurwitz spaces are available in the literature [31, 33–35] along with a candidate of Löwner-type equations [36].

Acknowledgments

The authors are partially supported by Grant-in-Aid for Scientific Research Nos 18340061, 18540210 and 19540179 from the Japan Society for the Promotion of Science.

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